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Boson-fermion Dyson mapping and supersymmetry in fermion systems

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Abstract

We demonstrate that exact supersymmetry can emerge in a purely fermionic system. This "supersymmetry without bosons" is unveiled by constructing a novel boson-fermion Dyson mapping from a fermion space to a space comprised of collective bosons and ideal fermions. In a nuclear structure context the collective spectra of even and odd nuclei can then be unified in a supersymmetric description with Pauli correlations still exactly taken into account.

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In its original inception supersymmetry pertains to a system of fermions and bosons exhibiting an invariance with respect to exchange between these two classes of particles. It may therefore be somewhat surprising to discover that a fermion system on its own can also exhibit *exact supersymmetry*. This is discussed in general below and demonstrated for a specific nuclear model.

The notion of supersymmetry has in fact proved to be fruitful in nuclear structure physics [1]. Properties of some neighboring even and odd nuclei can namely be classified and understood in terms of an assumed supersymmetry within the framework of the interacting boson-fermion model (IBFM) [2]. To appreciate how this may come about and why this phenomenological supersymmetry does not necessarily imply exact supersymmetry on the microscopic level, we first briefly recapitulate some basics of nuclear structure.

The atomic nucleus is composed of fermions whose interactions can be described quite successfully on a non-relativistic level by a fermion hamiltonian. States of even and odd nuclei are in principle eigenstates of the same hamiltonian obtained for different particle numbers. Although there is no fundamental difference between even and odd nuclei from this point of view, their properties are, however, quite different.

Even-even nuclei, in which numbers of protons and neutrons are both even, have 0^+ ground states, 2^+ first excited states in most cases and very often low-lying sequences of collective states. Even-odd nuclei, with an odd number of either neutrons or protons, have ground states characterized by the spin and parity of the odd particle, with the particle-core interaction and core deformation playing a decisive role.

A unification of spectra of even and odd nuclei into a single framework is therefore a challenging possibility, with the prospect of unveiling a basic underlying symmetry. On the phenomenological level the IBFM [2] achieves such a unification. Starting from a common boson-fermion hamiltonian one finds that in some instances states of an even nucleus, described by many-boson wave functions, are linked by supersymmetry to states in a neighboring odd nucleus in which the odd fermion is treated explicitly. It is important to realize, however, that Pauli correlations between the odd particle and the even core are not fully taken into account in the IBFM. In this sense the link between the observed supersymmetry and the underlying microscopy is tenuous and no microscopic derivation of the IBFM hypothesis in fact exists so far.

In the present paper we systematically derive a supersymmetric nuclear hamiltonian starting from a purely fermionic microscopic collective model, while incorporating all Pauli correlations exactly. We assume a hamiltonian which is a quadratic scalar function of collective fermion pair creation operators $A^j{=}\frac{1}{2}\chi^j_{\mu\nu}a^\mu a^\nu$, pair annihilation operators A_i, and their commutators $[A_i, A^j]$. The notation exploits a summation convention in which upper (lower) indices denote creation (annihilation) operators, as in $a^\mu{=}a^+_\mu$ and $A^j{=}A^+_j$. We also assume [3,4] that the collective fermion operators obey the algebraic closure relations $[[A_i, A^j]\,, A_k] = c^{jl}_{ik}A_l$, i.e., operators A^j, A_i, and $[A_i, A^j]$, to form a collective spectrum generating algebra [5]. These relations guarantee an exact decoupling of the even collective space $|\Psi_{\rm even}\rangle = A^iA^j \dots A^k|0\rangle$ from all other even fermion states. Similarly, by adding an odd fermion, one obtains an exactly decoupled odd collective space $|\Psi_{\rm odd}\rangle = a^\mu A^iA^j \dots A^k|0\rangle$.

In a supersymmetric description, or any boson-fermion description for that matter, an even state $|\Psi_{\rm even}\rangle$ should be represented by an ideal boson space state $|\Psi_{\rm even})$, say, with the odd ideal space states $|\Psi_{\rm odd})$ containing an additional fermion α^μ. Following traditional

terminology this is also referred to as an ideal fermion [6]. By definition it commutes with all boson operators, $[\alpha^\mu, B_i]$=$[\alpha^\mu, B^i]$=0. The required representation in the ideal space can be achieved by constructing an appropriate boson-fermion mapping. In Ref. [4] supercoherent states have been used to derive the boson-fermion mapping,

$$A^j \longleftrightarrow R^j \equiv \mathcal{A}^j + B^i[\mathcal{A}_i, \mathcal{A}^j] - \tfrac{1}{2}c^{jl}_{ik}B^iB^kB_l, \tag{1a}$$

$$[A_i,A^j] \longleftrightarrow [\mathcal{A}_i, \mathcal{A}^j] - c^{jl}_{ik}B^kB_l, \tag{1b}$$

$$A_j \longleftrightarrow B_j, \tag{1c}$$

$$a^\nu \longleftrightarrow \alpha^\nu + B^i[\mathcal{A}_i, \alpha^\nu], \tag{1d}$$

$$a_\nu \longleftrightarrow \alpha_\nu, \tag{1e}$$

where $\mathcal{A}^j$=$\frac{1}{2}\chi^j_{\mu\nu}\alpha^\mu\alpha^\nu$ are collective pairs of *ideal fermions*. This mapping exactly preserves the commutation relations of the collective algebra, as well as the commutation relations between single-fermion and pair operators, and also the anticommutation relations between single-fermion operators.

The mapping of states which results from the above mapping of operators is, however, not satisfactory [4]. Even states in particular are not entirely bosonized and two-fermion collective states are e.g. mapped as $A^j|0\rangle \longleftrightarrow (gB^j+\mathcal{A}^j)|0)$, where for $\chi_i^{\mu\nu} = (\chi^i_{\mu\nu})^*$, $\frac{1}{2}\chi_i^{\mu\nu}\chi^j_{\mu\nu} = g\delta^j_i$ defines the normalization factor g. In fact, by using ideal fermions in addition to bosons one has introduced a large redundancy in the boson-fermion space. However, this freedom can be used to map two-fermion states $A^j|0\rangle$ on an arbitrary linear combination of the one-boson states $B^j|0)$ and one-ideal-fermion-pair states $\mathcal{A}^j|0)$. A mapping free of the above redundancy can thus be found and constructed by applying to mapping (1) a suitable similarity transformation.

Some examples of appropriate similarity transformations have been presented in Ref. [4]. They led, however, to infinite series of operators and were rather impractical in applications. Here we derive a similarity transformation which does not have this disadvantage. The key point of the derivation is the observation that by simply disregarding the ideal-fermion pair $\mathcal{A}^j$ in the mapping of the collective pair A^j, Eq. (1a), we still preserve the collective commutation relations. Therefore, there exists a similarity transformation X of the boson-fermion images (1) which gives the following mapping:

$$A^j \longleftrightarrow R^j - \mathcal{A}^j = B^i[\mathcal{A}_i, \mathcal{A}^j] - \tfrac{1}{2}c^{jl}_{ik}B^iB^kB_l, \tag{2a}$$

$$[A_i,A^j] \longleftrightarrow [\mathcal{A}_i, \mathcal{A}^j] - c^{jl}_{ik}B^kB_l, \tag{2b}$$

$$A_j \longleftrightarrow B_j, \tag{2c}$$

$$a^\nu \longleftrightarrow X^{-1}\left(\alpha^\nu + B^i[\mathcal{A}_i, \alpha^\nu]\right)X, \tag{2d}$$

$$a_\nu \longleftrightarrow X^{-1}\alpha_\nu X. \tag{2e}$$

Before discussing properties of this mapping and its relation to supersymmetry, we show that the similarity transformation X has the explicit form

$$X = \sum_{n=0}^{\infty}\Big(\frac{1}{C_F - \widehat{C}_F}\mathcal{A}^iB_i\Big)^n_{\frown} \quad , \tag{3}$$

where C_F=$\mathcal{A}^k\mathcal{A}_k$ is a Casimir operator of the ideal-fermion core subalgebra composed of operators $[\mathcal{A}_i, \mathcal{A}^j]$, i.e., $[C_F, [\mathcal{A}_i, \mathcal{A}^j]]$=0. Here we use the notation [7,8] of a deferred-action

operator $\widehat{C}_F$ which should be evaluated at the position indicated by "$\frown$". Equivalently, one can write Eq. (3) by using multiple sums over eigenstates of C_F, in which case $1/(C_F - \widehat{C}_F)$ can be replaced by typical energy denominators.

First, $X^{-1}B_jX{=}B_j$, because X explicitly commutes with the boson annihilation operator B_j, accounting for (2c). Second, we show that $X^{-1}R^jX{=}R^j{-}\mathcal{A}^j$ by proving the identity

$$\left[X, R^j - \mathcal{A}^j\right] = \mathcal{A}^j X. \tag{4}$$

This is achieved by splitting X into an infinite sum of terms which individually increase the number of ideal fermions by 0, 2, 4,. . ., i.e., $X{=}\sum_{n=0}^{\infty} X_n$. As $R^j{-}\mathcal{A}^j$ must conserve the number of ideal fermions, Eq. (4) becomes the recurrence relation $[X_{n+1}, R^j - \mathcal{A}^j] = \mathcal{A}^j X_n$ which allows all X_n to be determined, if X_0 is known.

We are free to choose X_0 as an arbitrary operator which commutes with $R^j{-}\mathcal{A}^j$, and Eq. (3) reflects the simplest choice $X_0{=}1$. Then the term X_1 in Eq. (3) is a solution of the recurrence relation for $n{=}0$. Indeed we have

$$\left[\frac{1}{C_F - \widehat{C}_F}\mathcal{A}^i B_i{}_{\frown}, R^j - \mathcal{A}^j\right] = \frac{1}{C_F - \widehat{C}_F}\left[\mathcal{A}^i B_i, R^j - \mathcal{A}^j\right]_{\frown} = \frac{1}{C_F - \widehat{C}_F}(C_F\mathcal{A}^j - \mathcal{A}^j C_F)_{\frown} = \mathcal{A}^j, \tag{5}$$

where the first equality results from the fact that C_F commutes with $R^j{-}\mathcal{A}^j$ and the second one from the explicit calculation of the commutator. Similarly, we prove by induction that the X_{n+1} term of Eq. (3) is a solution of the recurrence relation provided the same is true for X_n:

$$\begin{aligned}[X_{n+1}, R^j - \mathcal{A}^j] &= \frac{1}{C_F - \widehat{C}_F}\left[\mathcal{A}^i B_i X_n, R^j - \mathcal{A}^j\right]_{\frown} = \frac{1}{C_F - \widehat{C}_F}\Big(\mathcal{A}^i B_i \mathcal{A}^j X_{n-1} + (C_F\mathcal{A}^j - \mathcal{A}^j C_F)X_n\Big)_{\frown} \\ &= {}_{\smile}\,\mathcal{A}^j \frac{1}{\breve{C}_F - \widehat{C}_F}(1 + \frac{\breve{C}_F - C_F}{C_F - \widehat{C}_F})\mathcal{A}^i B_i X_{n-1}{}_{\frown} = \mathcal{A}^j X_n\end{aligned} \tag{6}$$

where the operator $\breve{C}_F$ is in turn evaluated at "$\smile$".

We can now discuss the structure of images of single fermion operators, Eqs. (2d) and (2e). An explicit general evaluation these images is difficult because the operators α^ν and α_ν do not commute with the ideal fermion Casimir operator C_F and one has to consider branching rules of the collective algebra. Before presenting solutions in particular cases we can, however, analyze some general properties of these images.

We note that the similarity transformation (3) does not change any state in which there are no bosons B^j, and therefore the single-fermion states are mapped onto single ideal fermion states, $a^\nu|0\rangle \leftrightarrow \alpha^\nu|0)$. Since the images of pair creation operators (2a) do not change the ideal fermion number, collective odd states $|\Psi_{\rm odd}\rangle$ will be mapped onto ideal states with one ideal fermion only. Especially from a physical point of view, this result is a clear improvement over the solutions found in Ref. [4], where such odd states were mapped onto ideal states with many-fermion components.

Moreover, the two-fermion states are mapped as $a^\mu a^\nu|0\rangle \longleftrightarrow (\alpha^\mu\alpha^\nu + \chi_i^{\mu\nu}B^i - \frac{1}{C_F}\chi_i^{\mu\nu}\mathcal{A}^i)|0)$ and in general contain the non-collective pair of ideal fermions $\alpha^\mu\alpha^\nu|0)$. However, when the collective pair A^j is formed by summing the pairs $a^\mu a^\nu$ with collective amplitudes $\chi^j_{\mu\nu}$, the ideal non-collective pairs above recombine (note that $C_F\mathcal{A}^j|0){=}g\mathcal{A}^j|0)$)

and only the boson state $gB^j|0)$ remains. Again, since the images (2a) conserve the ideal fermion number the same recombination mechanism is also valid for any even state.

When the Casimir operator C_F of a spectrum generating algebra depends on the ideal fermion number operator only, one may derive an explicit form of the similarity transformation [9]. Here we only present the solution for the complete so($2N$) algebra of all pairs $a^\mu a^\nu$ for $\mu<\nu$,

$$X = \frac{(n+\hat{n}+1)!!}{(2\hat{n}+1)!!} \exp\left(\tfrac{1}{2}\alpha^\mu\alpha^\nu B_{\mu\nu}\right)_{\frown} , \tag{7}$$

where $n=\alpha^\mu\alpha_\mu$. This resembles the similarity transformations considered in Ref. [4] with the important modification of number-operator dependent expansion coefficients. The boson-fermion mapping now reads

$$\begin{aligned}
a^\mu a^\nu &\longleftrightarrow B^{\mu\nu} - B^{\mu\rho}B^{\nu\theta}B_{\rho\theta} - B^{\mu\rho}\alpha^\nu\alpha_\rho + B^{\nu\rho}\alpha^\mu\alpha_\rho, \\
a^\mu a_\nu &\longleftrightarrow B^{\mu\theta}B_{\nu\theta} + \alpha^\mu\alpha_\nu, \\
a_\nu a_\mu &\longleftrightarrow B_{\mu\nu}, \\
a^\nu &\longleftrightarrow \tfrac{n-2}{2n-3}\alpha^\nu + B^{\nu\rho}\alpha_\rho + \tfrac{1}{2n-1}\alpha^\tau B^{\nu\rho}B_{\rho\tau} \\
&\quad -\tfrac{1}{(2n-1)(2n-3)}\alpha^\sigma\alpha^\tau\alpha_\rho B^{\nu\rho}B_{\sigma\tau}, \\
a_\nu &\longleftrightarrow \alpha_\nu - \tfrac{1}{(2n-1)(2n-3)}\alpha^\sigma\alpha^\tau\alpha_\nu B_{\sigma\tau} + \tfrac{1}{2n-1}\alpha^\tau B_{\nu\tau}.
\end{aligned} \tag{8}$$

The mapping of the bifermion operators is identical to the one derived by Dönau and Janssen [10], while the mapping of the single-fermion operators completes their result preserving all fermion (anti-)commutation relations.

Clearly the above mapping (or similar mappings for a collective algebra [9]) will map a 1-plus-2-body hamiltonian onto a boson-fermion hamiltonian with the boson-fermion interaction expressible in terms of the supergenerators $B^\dagger\alpha$ and $\alpha^\dagger B$. In the ideal space one can then expect boson-fermion symmetry and under favorable circumstances even dynamical supersymmetry as we show in more detail for the boson-fermion mapping of an SO(8) model [11], defined by collective monopole and quadrupole fermion pairs S^+ and D^+.

In order to generalize the model to odd systems we include $2\Omega=(2i+1)(2k+1)$ creation and annihilation operators $a^\dagger_{jm}$ and a_{jm}, where $j=|k-i|,\ldots,k+i$ for an integer value of the inactive angular momentum k and the active angular momentum $i=3/2$ [11]. A boson-fermion mapping of this algebra can be derived from supercoherent states similarly to Eqs. (1) [9]. In this case the series defining the similarity transformation (3) cannot be explicitly summed up because the denominator $C_F - \hat{C}_F$, with $C_F = \frac{1}{2}n(\Omega+6-\frac{1}{2}n) - C_{2\mathrm{Spin_F}(6)}$, contains the quadratic Casimir operator of the Spin(6) group which is not expressible in terms of number operators.

A system of collective fermion pairs is now mapped onto a system of $s^\dagger$ and $d^\dagger$ bosons, and a system of collective fermion pairs with an odd fermion onto $s^\dagger$ and $d^\dagger$ bosons and one ideal fermion. This situation is reminiscent of the phenomenological IBFM and we proceed by showing that in the SO(8) model supersymmetry is in fact manifest in the mapped systems.

The six bosons above fix the boson sector of the supersymmetric structure in terms of U(6). The size of the fermion sector depends on k, and for $k=0$ ($j=3/2$) the four fermion states lead to an overall IBFM U(6/4) supersymmetry [12,13]. However, in SO(8) all particles

occupy the same j=3/2 level while in the IBFM it is assumed that the bosons occupy the whole valence shell with only the fermion restricted to j=3/2.

As a more realistic situation, consider k=2 corresponding to j=1/2, 3/2, 5/2, and 7/2. In the IBFM, a related supersymmetry with the same single-particle content is U(6/20), realized in the Au-Pt isotopes [14]. The group reduction chain is $\mathrm{U_B}(6)\otimes\mathrm{U_F}(20) \supset \mathrm{SO_B}(6)\otimes \mathrm{SU_F}(4) \supset \mathrm{Spin_{B+F}}(6) \supset \mathrm{Spin_{B+F}}(5) \supset \mathrm{Spin_{B+F}}(3)$. A Hamiltonian chosen as a linear combination of quadratic Casimir operators in the chain yields the analytic U(6/20) IBFM energy formula [14]

$$\begin{aligned} E &= A\sigma(\sigma+4) + \tilde{A}[\sigma_1(\sigma_1+4)+\sigma_2(\sigma_2+2)+\sigma_3^2] \\ &\quad + B[\tau_1(\tau_1+3)+\tau_2(\tau_2+1)] + CJ(J+1). \end{aligned} \tag{9}$$

In the SO(8) Ginocchio model all j=1/2, 3/2, 5/2, and 7/2 states are degenerate which leads to unrealistic odd spectra. One can lift this degeneracy by adding to the SO(8) algebra multipole operators corresponding to interchanged active and inactive angular momenta k and i. Suppose we add two such operators $\bar{P}_J$ for J=1 and 3, which form the SO(5) algebra and commute with all SO(8) generators. We may then consider the fermion group reduction chain $\mathrm{SO}(8)\otimes\mathrm{SO}(5) \supset \mathrm{Spin}(6)\otimes\mathrm{SO}(5) \supset \mathrm{Spin}(5)\otimes\mathrm{SO}(5) \supset \widetilde{\mathrm{Spin}}(5) \supset \widetilde{\mathrm{Spin}}(3)$. Here $\widetilde{\mathrm{Spin}}(5)$ is generated by $G^{(3)} = \sqrt{5}P_3 - 2\sqrt{2}\bar{P}_3$ and $G^{(1)} = \sqrt{5}P_1 + 2\sqrt{2}\bar{P}_1$, and $\widetilde{\mathrm{Spin}}(3)$ by $G^{(1)}$, where P_J are the original SO(8) multipole operators.

To arrive at an equivalent boson-fermion description we perform the SO(8) boson-fermion mapping discussed above, whereas the new SO(5) algebra is simply mapped from the original fermion space to the ideal fermion space. It can be shown that the boson-fermion images of generators of $\widetilde{\mathrm{Spin}}(5)$ are (up to a normalization factor) just the generators of the subgroup Spin(5) of U(6/20) in the IBFM [15]. Consequently, on the boson-fermion level we have now the following group chain $\mathrm{U_B}(6)\otimes\mathrm{U_F}(20) \supset \mathrm{U_B}(6)\otimes\mathrm{U_F}(4)\otimes\mathrm{U_F}(5) \supset \mathrm{SO_B}(6)\otimes\mathrm{SU_F}(4)\otimes \mathrm{SO_F}(5) \supset \mathrm{Spin_{B+F}}(6)\otimes\mathrm{SO_F}(5) \supset \mathrm{Spin_{B+F}}(5)\otimes\mathrm{SO_F}(5) \supset \widetilde{\mathrm{Spin}}_{\mathrm{B+F}}(5) \supset \widetilde{\mathrm{Spin}}_{\mathrm{B+F}}(3)$. The associated energy expression is then

$$\begin{aligned} E &= A[\sigma_1(\sigma_1+4)+\sigma_2(\sigma_2+2)+\sigma_3^2] \\ &\quad +B[\tau_1(\tau_1+3)+\tau_2(\tau_2+1)] \\ &\quad +\tilde{B}[\tilde{\tau}_1(\tilde{\tau}_1+3)+\tilde{\tau}_2(\tilde{\tau}_2+1)] + CJ(J+1), \end{aligned} \tag{10}$$

where (τ_1,τ_2) are the $\mathrm{Spin_{B+F}}(5)$ irreps with $\tau_2 = 1/2$ and $(\tilde{\tau}_1,\tilde{\tau}_2)$ are the irreps of $\widetilde{\mathrm{Spin}}_{\mathrm{B+F}}(5)$.

As a simple application of the supersymmetric $\mathrm{SO}(8)\otimes\mathrm{SO}(5)$ energy formula (10), we compare even and odd spectra in Fig. 1. The Hamiltonian parameters $B = 35$ keV, $\tilde{B} = 12.63$ keV, and $C = 18$ keV are chosen so that the even part coincides with the one of Ref. [14]. In the odd spectrum we obtain more low-lying J=3/2$^+$ states than one gets in the IBFM (9).

In summary, we have derived a new generalized Dyson boson-fermion mapping for a collective algebra extended by single fermion operators. The mapping gives finite non-hermitian boson-fermion images of collective pairs and single fermion operators expressed in terms of ideal boson and fermion annihilation and creation operators. As an application we have studied a fermionic model, namely $\mathrm{SO}(8)\otimes\mathrm{SO}(5)$, which extends the SO(8) model by a non-trivial interaction between the collective pairs and decoupled particles. In this model

we have revealed a supersymmetric structure analogous to the interacting boson-fermion model, but with full recognition of the Pauli principle.

Our study thus shows that supersymmetry, which in principle mixes bosonic and fermionic degrees of freedom, may equally well appear in purely fermionic models as it does in purely bosonic models, as recently demonstrated by Brzezinski *et al.* [16] and Plyushchay [17].

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FIGURES

FIG. 1. The lower part of an SO(8)$\otimes$SO(5) spectrum. In the odd part (right), the (1/2,1/2) and (3/2,1/2) irreps of Spin(5) are shown only. The dotted lines connect states belonging to the same $\widetilde{\mathrm{Spin}}(5)$ irrep.